DAY TWO

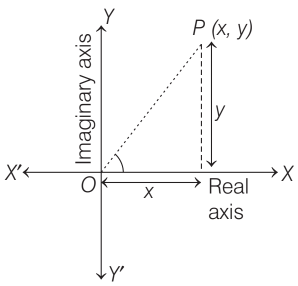
Complex Numbers

Learning & Revision for the Day

- Complex Numbers and Its Representation
- Algebra and Equality of Complex Numbers
- Conjugate and Modulus of a Complex Number
- Argument or Amplitude of a Complex Number
- Different forms of a Complex Number
- Concept of Rotation
- Square Root of a Complex Number
- De-Moivre's Theorem
- Cube Roots of Unity
- *n*th Roots of Unity
- Applications of Complex Numbers in Geometry

Complex Numbers and Its Representation

- A number in the form of $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, is called a **complex number**. The real numbers x and y are respectively called **real** and **imaginary** parts of complex number z.
  i.e. $x = \text{Re}(z)$, $y = \text{Im}(z)$ and the symbol i is called ***iota***.



Argand diagram

- A complex number $z = x + iy$ is said to be purely real if $y = 0$ and purely imaginary if $x = 0$.
- **Integral power of iota** (i)
  (i) $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$
  (ii) If n is an integer, then $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$ and $i^{4n+3} = -i$
  (iii) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$
- The complex number $z = x + iy$ can be represented by a point P in a plane called **argand plane** or **Gaussian plane** or **complex plane**. The coordinates of P are referred to the rectangular axes XOX' and YOY' which are called **real** and **imaginary axes**, respectively.

Algebra and Equality of Complex Numbers

If $z_1 = x_1 + i y_1$ and $z_2 = x_2 + i y_2$ are two complex numbers, then

(i) $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

(ii) $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

(iii) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

(iv) $\dfrac{z_1}{z_2} = \dfrac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$

(v) z_1 and z_2 are said to be equal if $x_1 = x_2$ and $y_1 = y_2$.

NOTE • Complex numbers does not possess any inequality, e.g. $3 + 2i > 1 + 2i$ does not make any sense.

Conjugate and Modulus of a Complex Number

- If $z = x + iy$ is a complex number, then **conjugate** of z is denoted by $\bar{z}$ and is obtained by replacing i by $-i$.
  i.e. $\bar{z} = x - iy$
- If $z = x + iy$, then **modulus** or **magnitude** of z is denoted by $|z|$ and is given by $|z| = \sqrt{x^2 + y^2}$

Results on Conjugate and Modulus

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}(z)$, $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z = \bar{z} \Leftrightarrow z$ is purely real.

(iv) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary.

(v) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(vi) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(vii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, if $z_2 \neq 0$

(viii) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where $a_i, b_i, c_i; (i = 1, 2, 3)$ are complex numbers.

(ix) $|z| = 0 \Leftrightarrow z = 0$

(x) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(xi) $-|z| \le \text{Re}(z), \text{Im}(z) \le |z|$

(xii) $|z_1 z_2| = |z_1||z_2|$

(xiii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $|z_2| \neq 0$

(xiv) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm \bar{z}_1 z_2 \pm z_1 \bar{z}_2$
$= |z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(z_1 \bar{z}_2)$

(xv) $|z^n| = |z|^n$, $n \in N$

(xvi) **Reciprocal of a complex number** For non-zero complex number $z = x + iy$, the reciprocal is given by $z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$.

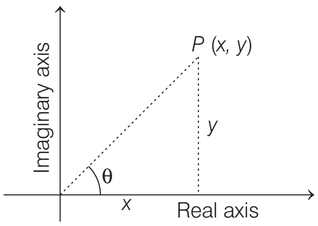
(xvii) **Triangle Inequality**
(a) $|z_1 + z_2| \le |z_1| + |z_2|$ (b) $|z_1 + z_2| \ge ||z_1| - |z_2||$
(c) $|z_1 - z_2| \le |z_1| + |z_2|$ (d) $|z_1 - z_2| \ge ||z_1| - |z_2||$

Argument or Amplitude of a Complex Number

Let $z = x + iy$ be a complex number, represented by a point $P(x, y)$ in the argand plane. Then, the angle θ which OP makes with the positive direction of Real axis (X-axis) is called the argument or amplitude of z and it is denoted by **arg (*z*)** or **amp (*z*)**.

The argument of z, is given by $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

- The value of argument θ which satisfies the inequality $-\pi < \theta \le \pi$, is called principal value of argument.
- The principal value of arg(z) is θ, $\pi - \theta$, $-\pi + \theta$ or $-\theta$ according as z lies in the 1st, 2nd, 3rd or 4th quadrants respectively, where $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$.



- Argument of z is not unique. General value of argument of z is $2n\pi + \theta$.

Results on Argument

If z, z_1 and z_2 are complex numbers, then

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(iii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(iv) The general value of $\arg(\bar{z})$ is $2n\pi - \arg(z)$.

(v) If z is purely imaginary then $\arg(z) = \pm\dfrac{\pi}{2}$.

(vi) If z is purely real then $\arg(z) = 0$ or π.

(vii) If $|z_1 + z_2| = |z_1 - z_2|$, then
$\arg\left(\dfrac{z_1}{z_2}\right)$ or $\arg(z_1) - \arg(z_2) = \dfrac{\pi}{2}$

(viii) If $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) = \arg(z_2)$

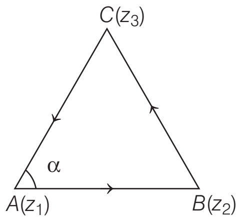
Different forms of a Complex Number

- **Polar or Trigonometrical Form** of $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$.
  If we use the general value of the argument θ, then the polar form of z is $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$, where n is an integer.
- **Euler's form** of $z = x + iy$ is $z = re^{i\theta}$, where $r = |z|, \theta = \arg(z)$ and $e^{i\theta} = \cos\theta + i\sin\theta$.

Concept of Rotation

Let z_1, z_2, z_3 be the vertices of ΔABC as shown in figure, then

$$\alpha = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \text{ and } \frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}$$



NOTE • Always mark the direction of arrow in anti-clockwise sense and keep that complex number in the numerator on which the arrow goes.

Square Root of a Complex Number

- If $z = a + ib$, then
$$\sqrt{z} = \sqrt{a + ib} = \pm \frac{1}{\sqrt{2}} [\sqrt{|z| + a} + i\sqrt{|z| - a}]$$
- If $z = a - ib$, then $\sqrt{z} = \sqrt{a - ib} = \pm \frac{1}{\sqrt{2}} [\sqrt{|z| + a} - i\sqrt{|z| - a}]$

De-Moivre's Theorem

- If n is any integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- If n is any rational number, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.
- If n is any positive integer, then
$$(\cos\theta + i\sin\theta)^{1/n} = \cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)$$
where, $k = 0, 1, 2, \dots n-1$

Cube Root of Unity

Cube roots of unity are $1, \omega, \omega^2$

where, $\omega = \frac{-1 + \sqrt{3}i}{2}$ and $\omega^2 = \frac{-1 - \sqrt{3}i}{2}$

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) $1 + \omega^n + \omega^{2n} = \begin{cases} 0 & \text{if } n \neq 3m, \quad m \in N \\ 3 & \text{if } n = 3m, \quad m \in N \end{cases}$

*n*th Roots of Unity

By nth root of unity we mean any complex number z which satisfies the equation $z^n = 1$.

(i) The nth roots of unity are $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$, where $\alpha = e^{\frac{i2\pi}{n}}$

(ii) $1 + \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{n-1} = 0$

(iii) $1 \cdot \alpha \cdot \alpha^2 \dots \alpha^{n-1} = [-1]^{n-1}$

Applications of Complex Numbers in Geometry

1. Distance between $A(z_1)$ and $B(z_2)$ is given by $AB = |z_2 - z_1|$.
2. Let point $P(z)$ divides the line segment joining $A(z_1)$ and $B(z_2)$ in the ratio $m : n$. Then,

   (i) for internal division, $z = \frac{mz_2 + nz_1}{m + n}$

   (ii) for external division, $z = \frac{mz_2 - nz_1}{m - n}$
3. Let ABC be a triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$, then centroid $G(z)$ of the ΔABC is given by z
$$= \frac{1}{3}(z_1 + z_2 + z_3)$$
   Area of ΔABC is given by $\Delta = \frac{1}{2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$
4. For an equilateral triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$, $z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$
5. The general equation of a straight line is $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.
6. (i) An equation of the circle with centre at z_0 and radius r, is $|z - z_0| = r$

   (ii) $|z - z_0| < r$ represents the interior of circle and $|z - z_0| > r$ represents the exterior of circle.

   (iii) General equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where b is real number, with centre is $-a$ and radius is $\sqrt{a\bar{a} - b}$.
7. If z_1 and z_2 are two fixed points and $k > 0, k \neq 1$ is a real number, then $\frac{|z - z_1|}{|z - z_2|} = k$ represents a circle.

   For $k = 1$, it represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.
8. If end points of diameter of a circle are $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on the circle, then equation of circle in diameter form is
$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 Real part of $\frac{1}{1-\cos\theta+i\sin\theta}$ is

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2}\tan\theta/2$ (d) 2

2 A value of θ, for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\sin^{-1}\frac{\sqrt{3}}{4}$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

3 $\sum_{n=1}^{13}(i^n+i^{n+1})$ is equal to

(a) i (b) $i-1$ (c) $-i$ (d) 0

4 If $\frac{z-1}{z+1}$ is a purely imaginary number (where, $z\neq-1$), then the value of $|z|$ is

(a) -1 (b) 1 (c) 2 (d) -2

5 If $z_1\neq0$ and z_2 are two complex numbers such that $\frac{z_2}{z_1}$ is a purely imaginary number, then $\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|$ is equal to → **JEE Mains 2013**

(a) 2 (b) 5 (c) 3 (d) 1

6 If $f(z)=\frac{7-z}{1-z^2}$, where $z=1+2i$, then $|f(z)|$ is equal to

(a) $\frac{|z|}{2}$ (b) $|z|$
(c) $2|z|$ (d) None of these

7 If $8iz^3+12z^2-18z+27i=0$, then the value of $|z|$ is

(a) $3/2$ (b) $2/3$ (c) 1 (d) $3/4$

8 If a complex number z satisfies the equation $z+\sqrt{2}|z+1|+i=0$, then $|z|$ is equal to → **JEE Mains 2013**

(a) 2 (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) 1

9 If α and β are two different complex numbers such that $|\alpha|=1, |\beta|=1$, then the expression $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$ is equal to

(a) $\frac{1}{2}$ (b) 1
(c) 2 (d) None of these

10 If $|z|=1$ and $\omega=\frac{z-1}{z+1}$ (where $z\neq-1$), then $\text{Re}(\omega)$ is

(a) 0 (b) $-\frac{1}{|z+1|^2}$
(c) $\frac{\sqrt{2}}{|z+1|^2}$ (d) None of these

11 If $\left|z-\frac{4}{z}\right|=2$, then the maximum value of $|z|$ is → **AIEEE 2009**

(a) $\sqrt{3}+1$ (b) $\sqrt{5}+1$ (c) 2 (d) $2+\sqrt{2}$

12 If z is a complex number such that $|z|\geq2$, then the minimum value of $\left|z+\frac{1}{2}\right|$ → **JEE Mains 2014**

(a) is equal to 5/2
(b) lies in the interval (1, 2)
(c) is strictly greater than 5/2
(d) is strictly greater than 3/2 but less than 5/2

13 If $|z_1|=2$, $|z_2|=3$ then $|z_1+z_2+5+12i|$ is less than or equal to

(a) 8 (b) 18 (c) 10 (d) 5

14 If $|z|<\sqrt{3}-1$, then $|z^2+2z\cos\alpha|$ is

(a) less than 2 (b) $\sqrt{3}+1$
(c) $\sqrt{3}-1$ (d) None of these

15 The number of complex numbers z such that $|z-1|=|z+1|=|z-i|$, is

(a) 0 (b) 1 (c) 2 (d) ∞

16 Number of solutions of the equation $|z|^2+7\bar{z}=0$ is/are

(a) 1 (b) 2 (c) 4 (d) 6

17 If $z\bar{z}+(3-4i)z+(3+4i)\bar{z}=0$ represent a circle, the area of the circle in square units is

(a) 5π (b) 10π (c) $25\pi^2$ (d) 25π

18 If $z=1+\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)$, then $\{\sin(\arg(z))\}$ is equal to

(a) $\sqrt{\frac{10-2\sqrt{5}}{4}}$ (b) $\frac{\sqrt{5}-1}{4}$
(c) $\frac{\sqrt{5}+1}{4}$ (d) None of these

19 If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals to → **JEE Mains 2013**

(a) $-\theta$ (b) $\frac{\pi}{2}-\theta$ (c) θ (d) $\pi-\theta$

20 Let z and ω are two non-zero complex numbers such that $|z|=|\omega|$ and $\arg z+\arg\omega=\pi$, then z equals

(a) $\bar{\omega}$ (b) ω
(c) $-\bar{\omega}$ (d) $-\omega$

21 If $|z-1|=1$, then $\arg(z)$ is equal to

(a) $\frac{1}{2}\arg(z)$ (b) $\frac{1}{3}\arg(z+1)$
(c) $\frac{1}{2}\arg(z-1)$ (d) None of these

22 Let $z=\cos\theta+i\sin\theta$. Then the value of $\sum_{m=1}^{15}\text{Im}\,(z^{2m-1})$ at, $\theta=2°$, is

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$ (c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

23 If $z=(i)^{(i)^{(i)}}$, where $i=\sqrt{-1}$, then $|z|$ is equal to

(a) 1 (b) $e^{-\pi/2}$ (c) 0 (d) $e^{\pi/2}$

24 $\left(\dfrac{1+i\sin\frac{\pi}{8}+\cos\frac{\pi}{8}}{1-i\sin\frac{\pi}{8}+\cos\frac{\pi}{8}}\right)^8$ equals to

(a) 2^8 (b) 0 (c) -1 (d) 1

25 If $1,\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$ are the nth roots of unity, then $(2-\alpha_1)(2-\alpha_2)\ldots(2-\alpha_{n-1})$ is equal to

(a) n (b) 2^n (c) 2^n+1 (d) 2^n-1

26 If $\omega(\neq 1)$ is a cube root of unity and $(1+\omega)^7=A+B\omega$. Then, (A, B) is equal to

(a) (1,1) (b) (1, 0) (c) (−1, 1) (d) (0, 1)

27 If $\alpha,\beta\in C$ are the distinct roots of the equation $x^2-x+1=0$, then $\alpha^{101}+\beta^{107}$ is equal to **➔ JEE Mains 2018**

(a) −1 (b) 0 (c) 1 (d) 2

28 If $x^2+x+1=0$, then $\sum_{r=1}^{25}\left(x^r+\frac{1}{x^r}\right)^2$ is equal to

(a) 25 (b) $25\,\omega$
(c) $25\,\omega^2$ (d) None of these

29 Let ω be a complex number such that $2\omega+1=z$, where $z=\sqrt{-3}$. If $\begin{vmatrix}1 & 1 & 1\\ 1 & -\omega^2-1 & \omega^2\\ 1 & \omega^2 & \omega^7\end{vmatrix}=3k$, then k is equal to **➔ JEE Mains 2017**

(a) $-z$ (b) z (c) -1 (d) 1

30 The value $\begin{vmatrix}1+\omega & \omega^2 & 1+\omega^2\\ -\omega & -(1+\omega^2) & (1+\omega)\\ -1 & -(1+\omega^2) & 1+\omega\end{vmatrix}$, where ω is cube root of unity, is equal to

(a) 2ω (b) $3\omega^2$ (c) $-3\omega^2$ (d) 3ω

31 If a, b and c are integers not all equal and ω is a cube root of unity (where, $\omega\neq 1$), then minimum value of $|a+b\omega+c\omega^2|$ is equal to

(a) 0 (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

32 Let $\omega=e^{i\pi/3}$, and a,b,c,x,y,z be non-zero complex numbers such that:

$$a+b+c=x;\ \ a+b\omega+c\omega^2=y;\ \ a+b\omega^2+c\omega=z$$

Then the value of $\dfrac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is:

(a) 1 (b) 2 (c) 3 (d) 4

33 If Re $\left(\frac{1}{z}\right)=3$, then z lies on

(a) circle with centre on Y-axis
(b) circle with centre on X-axis not passing through origin
(c) circle with centre on X-axis passing through origin
(d) None of the above

34 If the imaginary part of $(2z+1)/(iz+1)$ is −2, then the locus of the point representing z in the complex plane is

(a) a circle (b) a straight line
(c) a parabola (d) None of these

35 If $|z|=1$ and $z\neq\pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on

(a) a line not passing through the origin
(b) $|z|=\sqrt{2}$
(c) the X-axis
(d) the Y-axis

36 If $\omega=\dfrac{z}{z-\frac{i}{3}}$ and $|\omega|=1$, then z lies on

(a) a circle (b) an ellipse
(c) a parabola (d) a straight line

37 If z_1 and z_2 are two complex numbers such that $\frac{z_1}{z_2}+\frac{z_2}{z_1}=1$, then

(a) z_1,z_2 are collinear
(b) z_1,z_2 and the origin form a right angled triangle
(c) z_1,z_2 and the origin form an equilateral triangle
(d) None of the above

38 A complex number z is said to be unimodular, if $|z|=1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular.

Then, the point z_1 lies on a **➔ JEE Mains 2015**

(a) straight line parallel to X-axis
(b) straight line parallel to Y-axis
(c) circle of radius 2
(d) circle of radius $\sqrt{2}$

39 If $|z^2-1|=|z|^2+1$, then z lies on

(a) a real axis (b) an ellipse
(c) a circle (d) imaginary axis

40 Let z satisfy $|z|=1$ and $z=1-\bar{z}$ **➔ JEE Mains 2013**

Statement I z is a real number.

Statement II Principal argument of z is $\pi/3$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 For positive integers n_1 and n_2 the value of the expression $(1+i)^{n_1}+(1+i^3)^{n_1}+(1+i^5)^{n_2}+(1+i^7)^{n_2}$ where $i=\sqrt{-1}$, is a real number iff

(a) $n_1=n_2$ (b) $n_2=n_2-1$ (c) $n_1=n_2+1$ (d) $\forall\, n_1$ and n_2

2 If $z\neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies

(a) on the imaginary axis
(b) either on the real axis or on a circle passing through the origin
(c) on a circle with centre at the origin
(d) either on the real axis or on a circle not passing through the origin

3 Let ω be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying
$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$$ is equal to

(a) 0 (b) 1 (c) 2 (d) 4

4 The locus of $z=x+iy$ which satisfying the inequality $\log_{1/2}|z-1|>\log_{1/2}|z-i|$ is given by

(a) $x+y<0$ (b) $x-y>0$ (c) $x-y<0$ (d) $x+y>0$

5 Let $z_1=10+6i$, $z_2=4+6i$. If z is any complex number such that $\arg(z-z_1)/(z-z_2)=\pi/4$, then $|z-7-9i|$ is equal to

(a) 18 (b) $3\sqrt{2}$ (c) $3/\sqrt{2}$ (d) None of these

6 Let $z=x+iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3+\bar{z}z^3=350$ is

(a) 48 (b) 32 (c) 40 (d) 80

7 If $\alpha+i\beta=\cot^{-1}(z)$, where $z=x+iy$ and α is a constant, then the locus of z is

(a) $x^2+y^2-x\cot 2\alpha-1=0$
(b) $x^2+y^2-2x\cot\alpha-1=0$
(c) $x^2+y^2-2x\cot 2\alpha+1=0$
(d) $x^2+y^2-2x\cot 2\alpha-1=0$

8 If a complex number z lies in the interior or on the boundary of a circle of radius 3 and centre at $(-4, 0)$, then the greatest and least value of $|z+1|$ are

(a) 5, 0 (b) 6, 1 (c) 6, 0 (d) None of these

9 If z is any complex number satisfying $|z-3-2i|\le 2$, then the minimum value of $|2z-6+5i|$ is

(a) 2 (b) 3 (c) 5 (d) 6

10 A man walks a distance of 3 units from the origin towards the North-East (N 45° E) direction. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is

(a) $3e^{i\pi/4}+4i$ (b) $(3-4i)e^{i\pi/4}$
(c) $(4+3i)e^{i\pi/4}$ (d) $(3+4i)e^{i\pi/4}$

11 If $1, a_1, a_2 \ldots a_{n-1}$ are n^{th} roots of unity, then $\frac{1}{1-a_1}+\frac{1}{1-a_2}+\ldots+\frac{1}{1-a_{n-1}}$ equals to

(a) $\frac{2^n-1}{n}$ (b) $\frac{n-1}{2}$ (c) $\frac{n}{n-1}$ (d) None of these

12 For $z,\omega\in C$, if $|z|^2\omega-|\omega|^2z=z-\omega$, then z is equal to

(a) ω or $\bar{\omega}$ (b) ω or $\omega/|\omega|^2$
(c) $\bar{\omega}$ or $\omega/|\omega|^2$ (d) None of these

13 The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)$ is

(a) 1 (b) -1 (c) $-i$ (d) i

14 Let z_1 and z_2 be roots of the equation $z^2+pz+q=0$, $p,q\in C$. Let A and B represents z_1 and z_2 in the complex plane. If $\angle AOB=\alpha\neq 0$ and $OA=OB$; O is the origin, then $p^2/4q$ is equal to

(a) $\sin^2(\alpha/2)$ (b) $\tan^2(\alpha/2)$ (c) $\cos^2(\alpha/2)$ (d) None of these

15 If $1, \omega$ and ω^2 are the three cube roots of unity α, β, γ are the cube roots of $p, q<0$, then for any x, y, z the expression $\left(\frac{x\alpha+y\beta+z\gamma}{x\beta+y\gamma+z\alpha}\right)$ is equal to

(a) 1 (b) ω (c) ω^2 (d) None of these

ANSWERS

SESSION 1

1. (b)	**2.** (d)	**3.** (b)	**4.** (b)	**5.** (d)	**6.** (a)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (a)
11. (b)	**12.** (b)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (b)	**17.** (d)	**18.** (b)	**19.** (c)	**20.** (c)
21. (c)	**22.** (d)	**23.** (a)	**24.** (c)	**25.** (d)	**26.** (a)	**27.** (c)	**28.** (d)	**29.** (a)	**30.** (c)
31. (b)	**32.** (c)	**33.** (c)	**34.** (b)	**35.** (d)	**36.** (d)	**37.** (c)	**38.** (c)	**39.** (d)	**40.** (d)

SESSION 2

1. (d)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (d)	**8.** (c)	**9.** (c)	**10.** (d)
11. (b)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (c)					

Hints and Explanations

SESSION 1

1 Let $z = \dfrac{1}{1-\cos\theta + i\sin\theta}$

$= \dfrac{1}{2\sin^2(\theta/2) + 2i\sin(\theta/2)\cos(\theta/2)}$

$= \dfrac{1}{2i\sin(\theta/2)}\dfrac{1}{[\cos(\theta/2) - i\sin(\theta/2)]}$

$= \dfrac{\cos(\theta/2) + i\sin(\theta/2)}{2i\sin(\theta/2)} = \dfrac{1}{2} + \dfrac{1}{2i}\cot(\theta/2)$

$= \dfrac{1}{2} - i\cdot\dfrac{1}{2}\cot\theta/2$

2 Let $z = \dfrac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary

then we have

$\text{Re}(z) = 0$

Consider, $z = \dfrac{2+3i\sin\theta}{1-2i\sin\theta}$

$= \dfrac{(2+3i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$

$= \dfrac{(2-6\sin^2\theta)+(4\sin\theta+3\sin\theta)i}{1+4\sin^2\theta}$

$\because \quad \text{Re}(z) = 0$

$\therefore \quad \dfrac{2-6\sin^2\theta}{1+4\sin^2\theta} = 0$

$\Rightarrow \quad \sin^2\theta = \dfrac{1}{3} \Rightarrow \sin\theta = \pm\dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \pm\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

3 $\displaystyle\sum_{n=1}^{13}(i^n + i^{n+1}) = (1+i)\sum_{n=1}^{13} i^n$

$= (1+i)\dfrac{i(1-i^{13})}{1-i}$

$= i - 1 \quad [\because i^{13} = i, i^2 = -1]$

4 Let $z = x + iy$

$\dfrac{z-1}{z+1} = \dfrac{x+iy-1}{x+iy+1} = \dfrac{(x-1)+iy}{(x+1)+iy} \times \dfrac{(x+1)-iy}{(x+1)-iy}$

$= \dfrac{(x-1)(x+1) - iy(x-1) + iy(x+1) - i^2y^2}{(x+1)^2 - i^2y^2}$

$= \dfrac{x^2 - 1 + iy(x+1-x+1) + y^2}{(x+1)^2 + y^2}$

$\Rightarrow \dfrac{z-1}{z+1} = \dfrac{(x^2+y^2-1)}{(x+1)^2+y^2} + \dfrac{i(2y)}{(x+1)^2+y^2}$

Since, $\dfrac{z-1}{z+1}$ is purely imaginary.

$\therefore \quad \text{Re}\left(\dfrac{z-1}{z+1}\right) = 0$

$\Rightarrow \quad \dfrac{x^2+y^2-1}{(x+1)^2+y^2} = 0$

$\Rightarrow \quad x^2 + y^2 - 1 = 0$

$\Rightarrow \quad x^2 + y^2 = 1$

$\Rightarrow \quad |z|^2 = 1 \Rightarrow |z| = 1$

5 Given, $\dfrac{z_2}{z_1}$ is a purely imaginary

Let $z = ni$. Then,

$\left|\dfrac{2z_1+3z_2}{2z_1-3z_2}\right| = \left|\dfrac{2+3\cdot\dfrac{z_2}{z_1}}{2-3\cdot\dfrac{z_2}{z_1}}\right| = \dfrac{|2+3ni|}{|2-3ni|}$

$= \dfrac{\sqrt{4+9n^2}}{\sqrt{4+9n^2}} = 1$

6 Given, $f(z) = \dfrac{7-z}{1-z^2}$ and $z = 1+2i$

$\therefore \quad f(z) = \dfrac{7-(1+2i)}{1-(1+2i)^2}$

$= \dfrac{6-2i}{1-(1-4+4i)} = \dfrac{6-2i}{4-4i}$

$= \dfrac{6-2i}{4(1-i)} \times \dfrac{1+i}{(1+i)} = \dfrac{6+4i+2}{4(1^2-i^2)}$

$= \dfrac{8+4i}{4(2)} = \dfrac{1}{2}(2+i)$

Now, $|f(z)| = \dfrac{\sqrt{4+1}}{2} = \dfrac{\sqrt{5}}{2} = \dfrac{|z|}{2}$

$[\because z = 1+2i, \text{ given} \Rightarrow |z| = \sqrt{5}]$

7 Given, $8iz^3 + 12z^2 - 18z + 27i = 0$

$\Rightarrow \quad 4z^2(2iz+3) + 9i(2iz+3) = 0$

$\Rightarrow \quad (2iz+3)(4z^2+9i) = 0$

$\Rightarrow \quad 2iz + 3 = 0 \text{ or } 4z^2 + 9i = 0$

$\therefore \quad |z| = \dfrac{3}{2}$

8 We have, $(x+iy) + \sqrt{2}|x+iy+1| + i = 0$

[put $z = x+iy$]

$\Rightarrow \quad (x+iy) + \sqrt{2}\sqrt{(x+1)^2+y^2} + i = 0$

$\Rightarrow \quad x + \sqrt{2}\sqrt{(x+1)^2+y^2} = 0$

and $\quad y + 1 = 0$

$\Rightarrow \quad x + \sqrt{2}\sqrt{(x+1)^2+(-1)^2} = 0$

and $\quad y = -1$

$\Rightarrow \quad x^2 = 2[(x+1)^2+1]$

$\Rightarrow \quad x^2 = 2x^2 + 4x + 4$

$\Rightarrow \quad x^2 + 4x + 4 = 0 \Rightarrow (x+2)^2 = 0$

$\Rightarrow \quad x = -2$

$\therefore \quad z = -2 - i \Rightarrow |z| = \sqrt{4+1} = \sqrt{5}$

9 $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\dfrac{\beta-\alpha}{\beta\cdot\bar{\beta}-\bar{\alpha}\beta}\right|$

$\left[\begin{array}{l}\because \quad |\beta| = 1 \\ \text{and } |\beta|^2 = \beta\bar{\beta} = 1\end{array}\right]$

$= \left|\dfrac{\beta-\alpha}{\beta(\bar{\beta}-\bar{\alpha})}\right| = \dfrac{1}{|\beta|}\dfrac{|\beta-\alpha|}{|\bar{\beta}-\bar{\alpha}|} = \dfrac{|\beta-\alpha|}{|\beta-\alpha|} = 1$

$[\because |z| = |\bar{z}|]$

10 Given, $|z| = 1$

$\Rightarrow \quad z\bar{z} = 1$

Now, $2\text{Re}(\omega) = \omega + \bar{\omega} = \dfrac{z-1}{z+1} + \dfrac{\bar{z}-1}{\bar{z}+1}$

$= \dfrac{(z-1)(\bar{z}+1) + (\bar{z}-1)(z+1)}{|z+1|^2}$

$= \dfrac{2z\bar{z}-2}{|z+1|^2} = 0 \quad [\because z\bar{z} = 1]$

$\therefore \quad \text{Re}(\omega) = 0.$

11 $|z| = \left|\left(z - \dfrac{4}{z}\right) + \dfrac{4}{z}\right|$

$\Rightarrow \quad |z| \le \left|z - \dfrac{4}{z}\right| + \dfrac{4}{|z|}$

$\Rightarrow \quad |z| \le 2 + \dfrac{4}{|z|}$

$\Rightarrow \quad \dfrac{|z|^2 - 2|z| - 4}{|z|} \le 0$

Since, $|z| > 0$

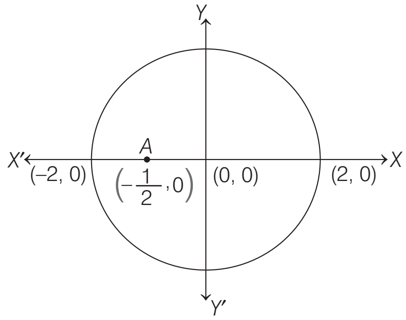
$\Rightarrow \quad |z|^2 - 2|z| - 4 \le 0$

$\Rightarrow [|z| - (\sqrt{5}+1)]\,[|z| - (1-\sqrt{5})] \le 0$

$\Rightarrow \quad 1 - \sqrt{5} \le |z| \le \sqrt{5} + 1$

12 $|z| \ge 2$ is the region on or outside circle whose centre is (0,0) and radius is 2.

Minimum $\left|z + \dfrac{1}{2}\right|$ is distance of z, which lie on circle $|z| = 2$ from $\left(-\dfrac{1}{2}, 0\right)$.



$\therefore$ Minimum $\left|z + \dfrac{1}{2}\right|$

$=$ Distance of $\left(-\dfrac{1}{2}, 0\right)$ from $(-2, 0)$

$= \sqrt{\left(-2+\dfrac{1}{2}\right)^2 + 0} = \dfrac{3}{2}$

Alternate Method

We know, $|z_1 + z_2| \ge ||z_1| - |z_2||$

$\therefore \quad \left|z + \dfrac{1}{2}\right| \ge \left||z| - \left|\dfrac{1}{2}\right|\right| = \left||z| - \dfrac{1}{2}\right|$

$\ge \left|z - \dfrac{1}{2}\right| = \dfrac{3}{2}$

$\therefore \quad \left|z+\frac{1}{2}\right| \geq \frac{3}{2}$

$\therefore$ Minimum value of $\left|z+\frac{1}{2}\right|$ is $\frac{3}{2}\cdot$

13 **Fact:** $|z_1 + z_2 + \ldots + z_n|$

$$\leq |z_1| + |z_2| + \ldots + |z_n|$$

$$\therefore \quad |z_1 + z_2 + (5+12i)|$$
$$\leq |z_1| + |z_2| + |5+12i|$$
$$= 2 + 3 + 13 = 18$$

14 Consider $|z^2 + 2z\cos\alpha| \leq |z|^2 + 2|z|$

$$|\cos\alpha| \leq |z|^2 + 2|z|$$
$$< (\sqrt{3}-1)^2 + 2(\sqrt{3}-1)$$
$$= 3 + 1 - 2\sqrt{3} + 2\sqrt{3} - 2 = 2$$

$\therefore \quad |z^2 + 2z\cos\alpha| < 2$

15 Let $z = x + iy$

$|z-1| = |z+1|$

$\text{Re } z = 0 \Rightarrow x = 0$

$|z-1| = |z-i| \Rightarrow x = y$

$|z+1| = |z-i| \Rightarrow y = -x$

Since, only (0, 0) will satisfy all conditions.

$\therefore$ Number of complex number $z = 1$.

16 **Given** $|z|^2 + 7\bar{z} = 0$

$\Rightarrow \quad z\bar{z} + 7\bar{z} = 0 \Rightarrow \bar{z}(z+7) = 0$

Case (i) : $\bar{z} = 0, \quad \therefore z = 0 = 0 + i0$

Case (ii) : $z = -7 \quad \therefore z = -7 + 0i$

Hence, there is only two solutions.

$z = 0$ and $z = -7$

17 Given $z\bar{z} + (3-4i)z + (3+4i)\bar{z} = 0$

Let $z = x + iy$

Then, $z\bar{z} = x^2 + y^2$

$$\therefore \quad x^2 + y^2 + (3-4i)(x+iy) + (3+4i)(x-iy) = 0$$
$$\Rightarrow \quad x^2 + y^2 + 6x + 8y = 0$$
$$\Rightarrow \quad (x^2 + 6x) + (y^2 + 8y) = 0$$
$$\Rightarrow \quad (x+3)^2 + (y+4)^2 = 3^2 + 4^2$$
$$\Rightarrow \quad [x-(-3)]^2 + [y-(-4)]^2 = 5^2$$

So, area of circle be $\pi R^2 = 25\pi$

$[\because R = \text{radius} = 5]$

18 If $z = 1 + \cos\theta + i\sin\theta$, then $\arg(z) = \frac{\theta}{2}$

$$\therefore \quad \arg(z) = \frac{\pi/5}{2} = \frac{\pi}{10}$$

$\Rightarrow \sin(\arg z)$

$$= \sin\left(\frac{\pi}{10}\right) = \sin 18° = \frac{\sqrt{5}-1}{4}$$

19 Given, $|z| = 1$ and $\arg z = \theta$

$\therefore \quad z = e^{i\theta}$ and $\bar{z} = \frac{1}{z}$

$$\text{Now, } \arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+\frac{1}{z}}\right)$$
$$= \arg(z) = \theta$$

20 Let $|z| = |\omega| = r$ and let $\arg\omega = \theta$

Then, $\omega = r(\cos\theta + i\sin\theta) = re^{i\theta}$

and $\arg z = \pi - \theta$

$$\text{Hence, } z = r(\cos(\pi-\theta) + i\sin(\pi-\theta))$$
$$= r(-\cos\theta + i\sin\theta)$$
$$= -r(\cos\theta - i\sin\theta)$$
$$z = -\bar{\omega}$$

21 Given, $|z-1| = 1 \Rightarrow z - 1 = e^{i\theta}$,

where $\arg(z-1) = \theta \quad \ldots$(i)

$\Rightarrow \quad z = e^{i\theta} + 1$

$\Rightarrow \quad z = 1 + \cos\theta + i\sin\theta$

$[\because e^{i\theta} = \cos\theta + i\sin\theta]$

$$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$$

$\Rightarrow \arg(z) = \frac{\theta}{2} = \frac{1}{2}\arg(z-1)$ [from Eq. (i)]

22 Given that $z = \cos\theta + i\sin\theta = e^{i\theta}$

$$\therefore \quad \sum_{m=1}^{15} \text{Im}(z^{2m-1}) = \sum_{m=1}^{15} \text{Im}(e^{i\theta})^{2m-1}$$
$$= \sum_{m=1}^{15} \text{Im}\, e^{i(2m-1)\theta}$$
$$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin 29\theta$$
$$= \frac{\sin\left(\theta + \frac{14\cdot 2\theta}{2}\right)\sin\left(\frac{15\cdot 2\theta}{2}\right)}{\sin\left(\frac{2\theta}{2}\right)}$$
$$= \frac{\sin(15\theta)\sin(15\theta)}{\sin\theta} = \frac{1}{4\sin 2°} \quad [\because \theta = 2°]$$

23 Clearly, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

$$\therefore \quad (i)^i = (e^{i\pi/2})^i = e^{i^2\cdot\frac{\pi}{2}} = e^{-\pi/2}$$

Now, $(i)^{(i)^{(i)}} = (i)^{e^{-\pi/2}} \Rightarrow z = (i)^{e^{-\pi/2}}$

$\Rightarrow \quad |z| = |i|^{e^{-\pi/2}} = 1$

24 Let $z = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$

Then, $\frac{1}{z} = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}$

$$\text{Now, } \left(\frac{1+\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}}{1+\cos\frac{\pi}{8}-i\sin\frac{\pi}{8}}\right)^8 = \left(\frac{1+z}{1+z^{-1}}\right)^8$$
$$= \left(\frac{(1+z)z}{(1+z)}\right)^8 = z^8 = \left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^8$$
$$= \cos 8\cdot\frac{\pi}{8} + i\sin 8\cdot\frac{\pi}{8}$$

[using De-moivre's theorem]

$= \cos\pi = -1 \, (\because \sin\pi = 0)$

25 Clearly, $(x-1)(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1}) = x^n - 1$

Putting $x = 2$, we get

$(2-\alpha_1)(2-\alpha_2)\ldots(2-\alpha_{n-1}) = 2^n - 1$

26 We have, $(1+\omega)^7 = A + B\omega$

We know that $1 + \omega + \omega^2 = 0$

$\therefore \quad 1 + \omega = -\omega^2$

$\Rightarrow \quad (-\omega^2)^7 = A + B\omega$

$\Rightarrow \quad -\omega^{14} = A + B\omega$

$\Rightarrow -\omega^2 = A + B\omega \Rightarrow 1 + \omega = A + B\omega$

$[\because \omega^{14} = \omega^{12}\cdot\omega^2 = \omega^2]$

On comparing both sides, we get

$A = 1, B = 1$

27 α, β are the roots of $x^2 - x + 1 = 0$

$\because$ Roots of $x^2 - x + 1 = 0$ are $-\omega, -\omega^2$

$\therefore \quad$ Let $\alpha = -\omega$ and $\beta = -\omega^2$

$$\Rightarrow \quad \alpha^{101} + \beta^{107} = (-\omega)^{101} + (-\omega^2)^{107}$$
$$= -(\omega^{101} + \omega^{214}) = -(\omega^2 + \omega)$$

$[\because \omega^{3n+2} = \omega^2$ and $\omega^{3n+1} = \omega]$

$= -(-1) = 1 \quad [1 + \omega + \omega^2 = 0]$

28 $x^2 + x + 1 = 0$

$\Rightarrow \quad x = \omega, \omega^2$

So, $x^r + \frac{1}{x^r} = \omega^r + \frac{1}{\omega^r} = -1$

or 2 according as r is not divisible by 3 or divisible by 3.

$\therefore$ Required sum

$$= 17(-1)^2 + 8\cdot 2^2 = 49$$

29 Given, $\quad z = 2\omega + 1$

$$\Rightarrow \omega = \frac{-1+z}{2} \Rightarrow \omega = \frac{-1+\sqrt{3}i}{2}$$

$[\because z = \sqrt{-3}]$

$\Rightarrow \omega$ is complex cube root of unity

$$\text{Now, } \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$\left[\begin{matrix}\because 1 + \omega + \omega^2 = 0 \\ \omega^7 = \omega\end{matrix}\right]$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 3 & 0 & 0 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$\Rightarrow \quad 3(\omega^2 - \omega^4) = 3k$$
$$\Rightarrow \quad k = \omega^2 - \omega \Rightarrow k = -1 - 2\omega$$
$$\Rightarrow \quad k = -(1+2\omega) \Rightarrow k = -z$$

30 Using $1 + \omega + \omega^2 = 0$, we get

$$\Delta = \begin{vmatrix} 1+\omega & \omega^2 & -\omega \\ 1+\omega^2 & \omega & -\omega^2 \\ \omega^2+\omega & \omega & -\omega^2 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$,

$$\Delta = \begin{vmatrix} 0 & \omega^2 & -\omega \\ 0 & \omega & -\omega^2 \\ \omega^2+2\omega & \omega & -\omega^2 \end{vmatrix}$$

$$= (\omega^2 + 2\omega)(-\omega + \omega^2) = -3\omega^2$$

31 $|a + b\omega + c\omega^2|^2 = (a + b\omega + c\omega^2)(a + b\bar{\omega} + c\bar{\omega}^2)$

$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$[\because \bar{\omega} = \omega^2 \text{ and } \bar{\omega}^2 = \omega]$

$= a^2 + b^2 + c^2 - ab - bc - ca$

$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

So, it has minimum value 1 for $a = b = 1$ and $c = 2$.

32 Clearly, $|x|^2 + |y|^2 + |z|^2 = x\bar{x} + y\bar{y} + z\bar{z}$

$= (a + b + c)(\bar{a} + \bar{b} + \bar{c})$

$+ (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\bar{\omega} + \bar{c}\bar{\omega}^2)$

$+ (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\bar{\omega}^2 + \bar{c}\bar{\omega})$

$= 3(|a|^2 + |b|^2 + |c|^2)$

$\Rightarrow \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$

33 Given, $\text{Re}\left(\frac{1}{z}\right) = 3 \Rightarrow \text{Re}\left(\frac{\bar{z}}{|z|^2}\right) = 3$

$\left[\because \frac{1}{z} = \frac{\bar{z}}{|z|^2}\right]$

$\Rightarrow \frac{x}{x^2 + y^2} = 3 \Rightarrow 3x^2 + 3y^2 - x = 0$

So, it is a circle whose centre is on X-axis and passes through the origin.

34 $\frac{2z + 1}{iz + 1} = \frac{(2x + 1) + 2iy}{(1 - y) + ix}$

$= \frac{[(2x + 1) + 2iy] \cdot [(1 - y) - ix]}{(1 - y)^2 - i^2x^2}$

$= \frac{(2x - y + 1) - (2x^2 + 2y^2 + x - 2y)i}{1 + x^2 + y^2 - 2y}$

$\therefore$ Imaginary part

$= \frac{-(2x^2 + 2y^2 + x - 2y)}{1 + x^2 + y^2 - 2y} = -2$

$\Rightarrow x + 2y - 2 = 0$, which is a straight line.

35 Clearly, $\frac{z}{1 - z^2} = \frac{z}{z\bar{z} - z^2} = \frac{1}{\bar{z} - z}$, which is always imaginary.

36 $|\omega| = 1 \Rightarrow |z| = \left|z - \frac{i}{3}\right|$

It is the perpendicular bisector of the line segment joining $(0, 0)$ to $\left(0, \frac{1}{3}\right)$ i.e. the line $y = \frac{1}{6}$.

37 Given, $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1 \Rightarrow z_1^2 + z_2^2 = z_1 z_2$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_1 z_3 + z_2 z_3$, where $z_3 = 0$

So, z_1, z_2 and the origin form an equilateral triangle.

38 Given, z_2 is not unimodular i.e. $|z_2| \neq 1$ and $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular.

$\Rightarrow \left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$

$\Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$ $[\because z\bar{z} = |z|^2]$

$\Rightarrow |z_1|^2 + 4|z_2|^2 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2$

$= 4 + |z_1|^2|z_2|^2 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2$

$\Rightarrow (|z_2|^2 - 1)(|z_1|^2 - 4) = 0$

$\because |z_2| \neq 1$

$\therefore |z_1| = 2$

Let $z_1 = x + iy \Rightarrow x^2 + y^2 = (2)^2$

Point z_1 lies on a circle of radius 2.

39 Let $z = re^{i\theta}$

Then, $|r^2 e^{2i\theta} - 1| = r^2 + 1$

$\Rightarrow (r^2 \cos 2\theta - 1)^2 + (r^2 \sin 2\theta)^2 = (r^2 + 1)^2$

$\Rightarrow r^4 - 2r^2 \cos 2\theta + 1 = r^4 + 2r^2 + 1$

$\Rightarrow \cos 2\theta = -1 \Rightarrow \theta = \frac{\pi}{2}$

$\Rightarrow z = r\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = ir$

40 Let $z = x + iy$

Then, $x^2 + y^2 = 1$

and $x + iy = 1 - (x - iy)$

$\Rightarrow x^2 + y^2 = 1$ and $2x = 1 \Rightarrow x = \frac{1}{2}$

and $y = \pm\frac{\sqrt{3}}{2}$

$\therefore z = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Now, take, $z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \frac{\pi}{3}$

SESSION 2

1 Clearly, $(1 + i)^{n_1} + (1 + i^3)^{n_1} + (1 + i^5)^{n_2} + (1 + i^7)^{n_2}$

$= (1 + i)^{n_1} + (1 - i)^{n_1} + (1 + i)^{n_2} + (1 - i)^{n_2}$

$= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{n_1}$

$+ \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right]^{n_1}$

$+ \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right]^{n_2}$

$+ \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{n_2}$

$= (\sqrt{2})^{n_1}\left[\cos\frac{n_1\pi}{4} + i\sin\frac{n_1\pi}{4}\right]$

$+ (\sqrt{2})^{n_1}\left[\cos\frac{n_1\pi}{4} - i\sin i\frac{n_1\pi}{4}\right]$

$+ (\sqrt{2})^{n_2}\left[\cos\frac{n_2\pi}{4} - i\sin n_2\frac{\pi}{4}\right]$

$+ (\sqrt{2})^{n_2}\left[\cos\frac{n_2\pi}{4} + i\sin\frac{n_2\pi}{4}\right]$

$= (\sqrt{2})^{n_1}\left[2\cos\frac{n_1\pi}{4}\right] + (\sqrt{2})^{n_2}\left[2\cos\frac{n_2\pi}{4}\right]$

which is purely real $\forall\, n_1, n_2$.

2 Clearly, $\frac{z^2}{z - 1} = \frac{\bar{z}^2}{\bar{z} - 1}$

$\Rightarrow z\bar{z}z - z^2 = \bar{z}z\bar{z} - \bar{z}^2$

$\Rightarrow |z|^2(z - \bar{z}) - (z - \bar{z})(z + \bar{z}) = 0$

$\Rightarrow (z - \bar{z})(|z|^2 - (z + \bar{z})) = 0$

Either $z = \bar{z} \Rightarrow$ real axis

or $|z|^2 = z + \bar{z} \Rightarrow z\bar{z} - z - \bar{z} = 0$

i.e. $(x^2 + y^2 = 2x)$

represents a circle passing through origin.

3 $\omega = e^{2\pi i/3}$ = imaginary cube root of unity

$\therefore 1 + \omega + \omega^2 = 0$

Now, $\Delta = \begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}$

$= \begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}$

(applying $R_1 \to R_1 + R_2 + R_3$)

$= z\begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}$

$= z\{[(z + \omega^2)(z + \omega) - 1] + [\omega^2 - \omega(z + \omega)] + [\omega - \omega^2(z + \omega^2)]\}$

$= z\{z^2 + z(\omega + \omega^2) + \omega^3 - 1 - \omega z - \omega^2 z\} = z^3$

$\therefore \Delta = 0 \Rightarrow z^3 = 0 \Rightarrow z = 0$ is the only solution.

4 In the problem, base $= 1/2 \in (0, 1)$

$\therefore |z - 1| < |z - i| \Rightarrow |z - 1|^2 < |z - i|^2$

$\Rightarrow (z - 1)(\bar{z} - 1) < (z - i)(\bar{z} + i)$

$[\because |z|^2 = z\bar{z}]$

$\Rightarrow (1 + i)z + (1 - i)\bar{z} > 0$

$\Rightarrow (z + \bar{z}) + i(z - \bar{z}) > 0$

$\Rightarrow \left(\frac{z + \bar{z}}{2}\right) + i\left(\frac{z - \bar{z}}{2}\right) > 0$

$\Rightarrow \left(\frac{z + \bar{z}}{2}\right) - \left(\frac{z - \bar{z}}{2i}\right) > 0$

$\Rightarrow \text{Re}(z) - \text{Im}(z) > 0 \Rightarrow x - y > 0$

5 $\arg\left(\frac{z - (10 + 6i)}{z - (4 + 6i)}\right) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{y - 6}{x - 10} - \tan^{-1}\frac{y - 6}{x - 4} = \frac{\pi}{4}$

[take $z = x + iy$]

$$\Rightarrow \quad \frac{\frac{y-6}{x-10}-\frac{y-6}{x-4}}{1+\frac{(y-6)(y-6)}{(x-10)(x-4)}}=1$$

$\Rightarrow \quad x^2+y^2-14x-18y+112=0$

$\Rightarrow \quad (x-7)^2+(y-9)^2=18=(3\sqrt{2})^2$

$\Rightarrow \quad |z-(7+9i)|=3\sqrt{2}$

6 We have, $z\bar{z}=(z^2+\bar{z}^2)=350$

$\Rightarrow \quad 2(x^2+y^2)(x^2-y^2)=350$

$\Rightarrow \quad (x^2+y^2)(x^2-y^2)=175$

Since $x, y \in I$, the only possible case which gives integral solutions, is

$x^2+y^2=25 \quad \ldots$(i)

$x^2-y^2=7 \quad \ldots$(ii)

From Eqs. (i) and (ii) $x^2=16;\ y^2=9$

$\Rightarrow \quad x=\pm 4;\ \ y=\pm 3 \Rightarrow$ Area $=48$

7 We have, $\alpha+i\beta=\cot^{-1}(z)$

$\Rightarrow \quad \cot(\alpha+i\beta)=x+iy$

and $\cot(\alpha-i\beta)=x-iy$

Now, consider

$\cot 2\alpha=\cot[(\alpha+i\beta)+(\alpha-i\beta)]$

$$=\frac{\cot(\alpha+i\beta)\cdot\cot(\alpha-i\beta)-1}{\cot(\alpha+i\beta)+\cot(\alpha-i\beta)}$$

$$=\frac{(x^2+y^2-1)}{2x}$$

$\therefore \quad x^2+y^2-2x\cot 2\alpha-1=0$

8 Given, $|z+4|\le 3$

Now, $|z+1|=|z+4-3|$

$\le |z+4|+|3|\le 3+3=6$

Hence, greatest value of $|z+1|=6$

Since, least value of the modulus of a complex number is 0.

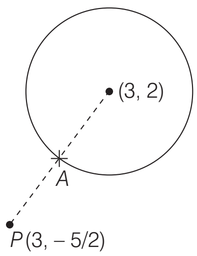
Consider, $|z+1|=0 \Rightarrow z=-1$

Now, $\quad |z+4|=|-1+4|=3$

$\Rightarrow |z+4|\le 3$ is satisfied by $z=-1$.

$\therefore$ Least value of $|z+1|=0$

9 $|z-3-2i|\le 2$



$\Rightarrow \quad z$ lies on or inside the circle

$(x-3)^2+(y-2)^2=2^2=4 \quad \ldots$ (i)

$$|2z-6+5i|=2\left|z-\left(3-\frac{5}{2}i\right)\right|$$

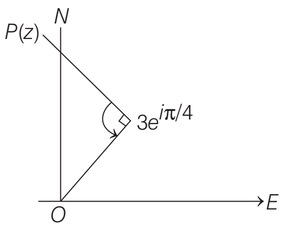
$$=2\left[\text{Distance of } z \text{ from } \left(3,-\frac{5}{2}\right)\right]$$

where z lies on circle (i).

$\therefore$ min $|2z-6+5i|=2PA=2\left(\frac{9}{2}-2\right)=5$

10 Clearly, $\dfrac{0-3e^{i\pi/4}}{z-3e^{i\pi/4}}=\dfrac{3}{4}e^{i\pi/2}$

$\therefore \quad \dfrac{-3e^{-i\pi/4}}{z-3e^{i\pi/4}}=\dfrac{3}{4}i$



$z-3e^{i\pi/4}=4ie^{i\pi/4}$

$z=(3+4i)e^{i\pi/4}$

11 Given $z^n=1$, where

$z=1, a_1, a_2, \ldots a_{n-1} \quad \ldots$(i)

Let $\alpha=\dfrac{1}{1-z}$, then $z=1-\dfrac{1}{\alpha}$

$\therefore \quad \left(1-\dfrac{1}{\alpha}\right)^n=1 \quad$ [by (i)]

$\Rightarrow \quad (\alpha-1)^n-\alpha^n=0$

$\Rightarrow \quad -C_1\alpha^{n-1}+C_2\alpha^{n-2}+\ldots+(-1)^n=0$

where, $\alpha=\dfrac{1}{1-a_1}, \dfrac{1}{1-a_2}, \ldots, \dfrac{1}{1-a_{n-1}}$

$$\Rightarrow \quad \frac{1}{1-a_1}+\frac{1}{1-a_2}+\ldots+\frac{1}{1-a_{n-1}}$$

$$=\frac{C_2}{C_1}=\frac{n(n-1)}{2/n}=\frac{(n-1)}{2}$$

12 $|z|^2\omega-|\omega|^2 z=z-\omega \quad \ldots$ (i)

$\Rightarrow \quad (|z|^2+1)\omega=(|\omega|^2+1)z$

$\Rightarrow \quad \dfrac{z}{\omega}=\dfrac{|z|^2+1}{|\omega|^2+1}=$ real

$\Rightarrow \quad \dfrac{z}{\omega}=\dfrac{\bar{z}}{\bar{\omega}} \Rightarrow \dfrac{z\bar{\omega}}{\omega}=\bar{z}\bar{\omega} \quad \ldots$ (ii)

Also, from Eq. (i), $z\bar{z}\omega-\omega\bar{\omega}z=z-\omega$

$\Rightarrow \quad z\bar{z}\omega-\omega\bar{z}\omega-z+\omega=0$

$\Rightarrow (\bar{z}\omega-1)(z-\omega)=0 \Rightarrow z=\omega$ or $\bar{z}\omega=1$

i.e. $\quad z\bar{\omega}=1$

$\Rightarrow \quad z=\omega$ or $z=\dfrac{1}{\bar{\omega}}=\omega/|\omega|^2$

13 We have, $\displaystyle\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)$

$$=i\sum_{k=1}^{10}\left(\cos\frac{2k\pi}{11}-i\sin\frac{2k\pi}{11}\right)$$

$$=i\sum_{k=1}^{10}e^{\frac{-i2k\pi}{11}}=i\sum_{k=1}^{10}\alpha^k$$

where $\alpha=e^{-i2\pi/11}$

$=i[\alpha+\alpha^2+\alpha^3+\ldots+\alpha^{10}]$

$$=i\frac{\alpha(1-\alpha^{10})}{1-\alpha}=i\frac{(\alpha-\alpha^{11})}{1-\alpha}$$

$$=i\frac{(\alpha-1)}{(1-\alpha)} \quad [\because \alpha^{11}=\cos 2\pi-\sin 2\pi=1]$$

$=-i$

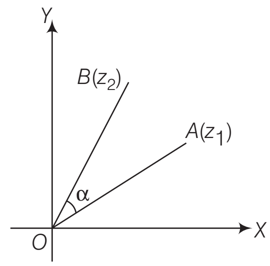
14 Given, $z^2+pz+q=0$

$\Rightarrow \quad z_1+z_2=-p$ and $z_1z_2=q$

$\because \quad OA=OB$

$\Rightarrow \quad |z_1|=|z_2|$

$\therefore \quad \dfrac{z_2}{z_1}=e^{i\alpha}=\cos\alpha+i\sin\alpha$



$\Rightarrow \quad \dfrac{z_1+z_2}{z_1}=1+\cos\alpha+i\sin\alpha$

$$=2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)$$

$\Rightarrow \quad \dfrac{(z_1+z_2)^2}{z_1^2}=4\cos^2\dfrac{\alpha}{2}e^{i\alpha}$

$$=4\cos^2\alpha/2\cdot\frac{z_2}{z_1}$$

$\Rightarrow \quad (z_1+z_2)^2=4\cos^2\dfrac{\alpha}{2}z_1z_2$

$\Rightarrow \quad p^2=4q\cos^2\alpha/2$

$\therefore \quad \dfrac{p^2}{4q}=\cos^2\dfrac{\alpha}{2}$

15 $\because p<0$, take $\quad p=-q^3(q>0)$

$\therefore \quad p^{1/3}=q(-1)^{1/3}=-q, -q\omega, -q\omega^2$

Now, take $\alpha=-q, \beta=-q\omega, \gamma=-q\omega^2$

Then, given expression

$$=\frac{x+y\omega+z\omega^2}{x\omega+y\omega^2+z}=\omega^2$$